2005 Georgia Tech High School Mathematics Competition

Junior-Varsity Proof Examination

SOLUTIONS

Problem 1: Consider a card-shuffling machine that permutes the order of its input. Suppose you put the following 13 cards into this machine in the following order:

$$A\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ J\ Q\ K$$

and then allow it to shuffle them twice, giving an output order of:

$$3\ 5\ 7\ 9\ J\ K\ Q\ 10\ 8\ 6\ 4\ 2\ A.$$

What was the order of the cards after the first shuffle?

If the machine permutes the cards, then after any multiple of 13 shuffles, we have the same order as originally put into the machine. We know how the machine permutes cards after two shuffles, so if we determine the circular mapping after 14 ($\equiv 1 \, mod \, 13$) shuffles, then we know the order after 1 shuffle.

$$P^2(\ A\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ J\ Q\ K\ ) = (\ 3\ 5\ 7\ 9\ J\ K\ Q\ 10\ 8\ 6\ 4\ 2\ A\ )$$
$$P^4 = P^2P^2 = P^2(\ 3\ 5\ 7\ 9\ J\ K\ Q\ 10\ 8\ 6\ 4\ 2\ A\ ) = (\ A\ 7\ 2\ J\ 9\ 10\ K\ 3\ Q\ 5\ 4\ 8\ 6\ )$$
$$P^8 = P^4P^4 = P^4(\ A\ 7\ 2\ J\ 9\ 10\ K\ 3\ Q\ 5\ 4\ 8\ 6\ ) = (\ A\ 2\ 9\ K\ Q\ 4\ 6\ 7\ J\ 10\ 3\ 5\ 8\ )$$
$$P^{12} = P^4P^8 = P^4(\ A\ 2\ 9\ K\ Q\ 4\ 6\ 7\ J\ 10\ 3\ 5\ 8\ ) = (\ A\ J\ K\ 5\ 6\ 2\ 10\ Q\ 8\ 7\ 9\ 3\ 4\ )$$
$$P^{14} = P^2P^{12} = P^2(\ A\ J\ K\ 5\ 6\ 2\ 10\ Q\ 8\ 7\ 9\ 3\ 4\ ) = (\ A\ 4\ 3\ 9\ 7\ 8\ Q\ 10\ 2\ 6\ 5\ K\ J\ ) = P^1$$

Therefore the order of the cards after one shuffle is:

$$(\ 4\ 6\ 9\ 3\ K\ 5\ 8\ Q\ 7\ 2\ A\ 10\ J\ )$$

Problem 2: The greatest common divisor (gcd) of two nonzero integers a and b is the largest of all common divisors of a and b. For example, $gcd(6, 8) = 2$, $gcd(9, 18) = 9$, and $gcd(7, 9) = 1$.

The least common multiple (lcm) of two nonzero integers a and b is the smallest of all common multiples of a and b. For example, $lcm(6, 8) = 24$, $lcm(9, 18) = 18$, and $lcm(7, 9) = 63$.

Is it true that $a \cdot b = lcm(a, b) \cdot gcd(a, b)$ for all positive integers a and b? If so, prove it. If not, present a case where it is false.

By the fundamental theorem of arithmetic we may write any number uniquely as the product of its prime factors. Then, $a = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n}$ and similarly, $b = P_1^{\beta_1} P_2^{\beta_2} \cdots P_n^{\beta_n}$, where α_i and β_i are nonnegative integers and the set $\{P_1, \ldots, P_n\}$ is the set containing all prime factors of a and b. That is, if one of the prime factors P_i is a factor of one but not the other, its corresponding exponent is 0. Example: $6 = 2^1 3^1$ and $4 = 2^2 3^0$.

Then, $gcd(a,b)$ can be written as

$$gcd(a,b) = gcd(\prod_{i=1}^{n} P_i^{\alpha_i}, \prod_{i=1}^{n} P_i^{\beta_i}) = \prod_{i=1}^{n} P_i^{min(\alpha_i,\beta_i)}.$$

Similarly, $lcm(a,b)$ can be written as

$$lcm(a,b) = lcm(\prod_{i=1}^{n} P_i^{\alpha_i}, \prod_{i=1}^{n} P_i^{\beta_i}) = \prod_{i=1}^{n} P_i^{max(\alpha_i,\beta_i)}.$$

Therefore,

$$gcd(a,b) \cdot lcm(a,b) = \prod_{i=1}^{n} P_i^{min(\alpha_i,\beta_i)+max(\alpha_i,\beta_i)} = \prod_{i=1}^{n} P_i^{\beta_i+\alpha_i} = a \cdot b.$$

Problem 3: In a certain school of 2005 students, each student is assigned a unique integer between 1 and 2005, inclusive, as their student number. The students are also assigned lockers of their own that bear their student number. Assume that all the locker doors are initially closed. If each student walks down the hall and opens/closes the door of every locker that is divisible by their student number, how many locker doors are open at the end of the day? Explain your answer.

For every locker door with number n, consider the prime factorization of $n = P_1^{\alpha_1} \ldots P_m^{\alpha_m}$ as above. The number of times that a door is toggled is equal to the total number of divisors, which is even unless n is a perfect square as follows:

Consider the pairing of every factor with its corresponding factor, if k divides n, then $\frac{n}{k}$ is an integer. The pair $(k, \frac{n}{k})$ represents two toggles unless $k = \frac{n}{k}$ or equivalently, n is a perfect square.

Finally, the number of doors that end up open is $\lfloor\sqrt{2005}\rfloor$=44.

Problem 4: How many unique ways could a person make change for a one-dollar bill using only combinations of the most commonly used coin denominations (1-cent, 5-cent, 10-cent, and 25-cent)?

We construct a set of polynomials such that when multiplied, the coefficient of the t^k term tells us how many ways there are to create change for k-cents. Consider:

$$p(t) = penny(t) = 1 + t + t^2 + \ldots + t^n + \ldots \qquad n(t) = nickel(t) = 1 + t^5 + t^{10} + \ldots$$
$$d(t) = dime(t) = 1 + t^{10} + t^{20} + \ldots \qquad q(t) = quarter(t) = 1 + t^{25} + t^{50} + \ldots$$

First note that if we ever used a penny, we would need to use 5 of them since no other coin would be able to make up to the next multiple of \$0.05 and \$1.00 is a multiple of \$0.05. This allows us to reduce $penny(t)$ to be identical to $nickel(t)$. Second, we are only interested in polynomial terms of degree $k \leq 100$; any exponent high than 100 represents a value higher than \$1.00 and can be freely neglected.

Therefore, to find the number of ways to make \$1.00 simply look at the coefficient of t^{100} in the product $n(t)^2 \cdot d(t) \cdot q(t)$. This can be done either combinatorially or simply by finding the product directly. Since multiplying these equations together increases the exponent very quickly, we choose direct manipulation as the most straightforward.

$$p(t) \cdot n(t) = n^2(t) = 1 + 2t^5 + 3t^{10} + 4t^{15} + ... + (n+1)t^{5n} + ...$$

$$d(t) \cdot q(t) = 1 + t^{10} + t^{20} + t^{25} + t^{30} + t^{35} + t^{40} + t^{45} + 2t^{50} + t^{55} + 2t^{60} + t^{65} + 2t^{70} + 2t^{75} + 2t^{80} + 2t^{85} + 2t^{90} + 2t^{95} + 3t^{100}$$

We only want to know the coefficient of t^{100}, so we add the possible combinations of t^{100} without calculating the entire product.

$$\text{Coefficient}(t^{100}) = 1 \cdot 21 + 1 \cdot 19 + 1 \cdot 17 + 1 \cdot 16 + 1 \cdot 15 + 1 \cdot 14 + 1 \cdot 13 + 1 \cdot 12 + 2 \cdot 11 + 1 \cdot 10 + 2 \cdot 9 + 1 \cdot 8$$
$$+ 2 \cdot 7 + 2 \cdot 6 + 2 \cdot 5 + 2 \cdot 4 + 2 \cdot 3 + 2 \cdot 2 + 3 \cdot 1) = 242$$

Problem 5: Prove that $2^{p-1} \equiv 1 \, mod \, p$, where $p \geq 3$ is any prime number.

Consider the term 2^p expanded using the binomial theorem:

$$2^p = (1+1)^p = \sum_{k=0}^{p} \binom{p}{k} 1^k = \sum_{k=0}^{p} \binom{p}{k} = \binom{p}{0} + \binom{p}{1} + ... + \binom{p}{p-1} + \binom{p}{p}$$

Each of the middle terms is divisible by p,

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = p \cdot \frac{(p-1)!}{k!(p-k)!} \equiv 0 \, mod \, p$$

while the end terms are equal to one. Therefore, the sum reduces to $2^p \equiv 2 \, mod \, p$. Applying the inverse of 2 to both sides and we see that $2^{p-1} \equiv 1 \, mod \, p$ for all prime $p \geq 3$.