2004 Georgia Tech High School Mathematics Competition
Varsity Proof Solution

v#1. Note that the total length of $RA + PA + AQ = 2(RA + \frac{1}{2}AQ)$.

Claim: The minimum value of the total length is achieved when $\angle RAP = \angle PAQ = \angle QAR = 120°$. In other words, $\angle RAO = 60°$.

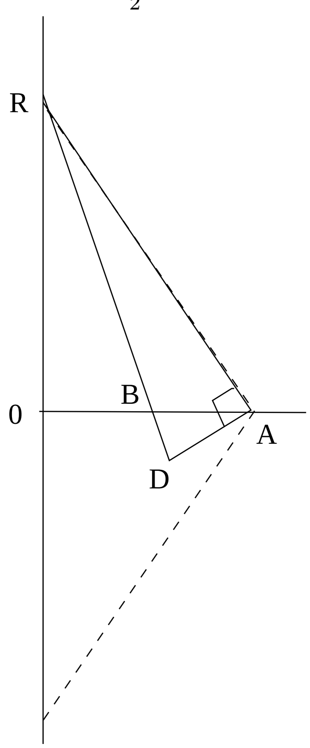
To prove the claim, we compare the length $RB + \frac{1}{2}BA$ and RA (resp. $RC + \frac{1}{2}AC$ and RA) if B lies to the left of A (resp. C lies to the right of A). If $\angle RAD = 90°$, then $\angle BAD = 30°$ so $\frac{1}{2}AB = BD$. But $\angle RAD = 90°$ implies that

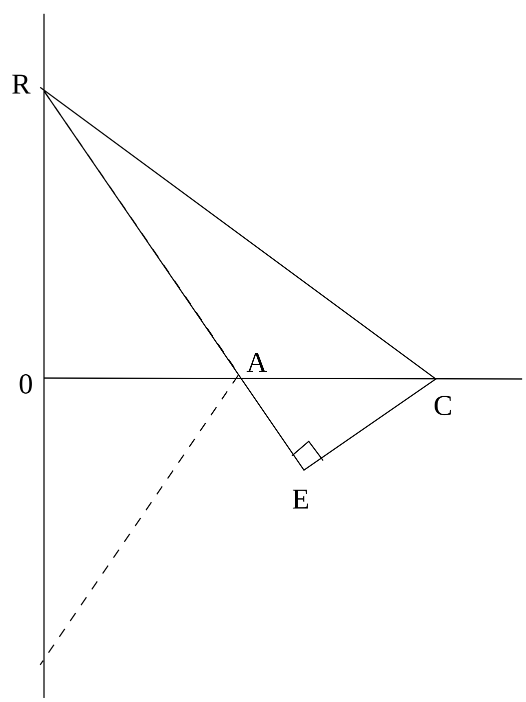
$$AR < RD = RB + BD = RB + \frac{1}{2}AB.$$

On the other hand, $\frac{1}{2}AC = AE$ and same reason as before,

$$RC > RE = RA + AE = RA + \frac{1}{2}AC$$

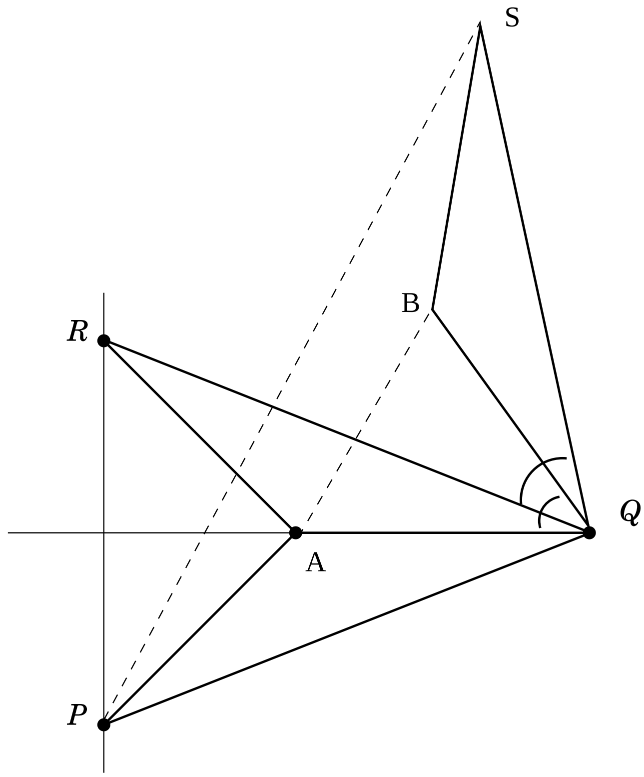
or $RC - \frac{1}{2}AC > RA$. So the claim is proved.





Alternate Solution to v#1

v#1. Revolving about Q clockwise by $\pi/3$ (60 degrees), from R and A one gets S and B respectively. Then $AR = BS$, and $AQ = AB$ since ABQ is an equilateral triangle. The sum $PA + AQ + AR$ is equal to the length of the zigzag line $PABS$, which is greater than or equal to the straight segment PS, with equality reached when the angle PAR is 120 degrees.



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v#2. First we claim at most one point with both integer coordinates is on a circle

$$C : \left(x - \sqrt{2}\right)^2 + \left(y - \sqrt{3}\right)^2 = r^2.$$

Suppose $(x_1, y_1), (x_2, y_2)$ are on C with x_1, x_2, y_1, y_2 are integers. Then we have

$$\begin{cases} \left(x_1 - \sqrt{2}\right)^2 + \left(y_1 - \sqrt{3}\right)^2 = r^2 \\ \left(x_2 - \sqrt{2}\right)^2 + \left(y_2 - \sqrt{3}\right)^2 = r^2. \end{cases}$$

Subtract the two equations, we get

$$\left(x_1 + x_2 - 2\sqrt{2}\right)(x_1 - x_2) + \left(y_1 + y_2 - 2\sqrt{3}\right)(y_1 - y_2) = 0,$$

or $k\left(\ell - 2\sqrt{2}\right) + m\left(n - 2\sqrt{3}\right) = 0$ where $k = x_1 - x_2$, $m = y_1 - y_2$, k, m, ℓ, n are integers. Since $\sqrt{2}$ and $\sqrt{3}$ are irrationals, so we must have $k\sqrt{2} - m\sqrt{3} = 0$. But this is impossible unless $k = m = 0$. Thus, $x_1 = x_2$, $y_1 = y_2$, or C passes at most one point with both coordinates being integers.

As r increases, inside C, an additional integer lattice point will be gained one at a time and we have proved the statement in the problem.

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v#3. Before proving the existence of infinitely many pairs (x, y) we first look at possible candidates. Since $x(x+1) \mid y(y+1)$ but $x, x+1 \nmid y, y+1$, we know that neither x nor $x+1$ can be a prime. Furthermore, by the fact that $x, x+1$ are coprime, and $y, y+1$ are coprime we actually know x or $x+1$ cannot be a prime power. This is because if, say, $x = p^k$ for some prime p, then $p \mid y(y+1)$. So $p \mid y$ or $p \mid y+1$. But $y, y+1$ are coprime, so $p^k \mid y$ if $p \mid y$ and $p^k \mid y+1$ if $p \mid y+1$. This contradicts the conditions $x \nmid y$, $x \nmid y+1$.

The smallest x such that neither x nor $x+1$ is a prime power is $x = 14$. We now construct infinitely many y such that $(14, y)$ is an eligible pair. Note $x(x+1) = 14 \times 15 = 2 \times 7 \times 3 \times 5$. If we can find y such that $2 \times 5 \mid y$, $7 \times 3 \mid y+1$, then we'll have $x(x+1) \mid y(y+1)$. Furthermore, the fact that $y, y+1$ are coprime implies that (x, y) is an eligible pair (why? check it!).

The rest is easy. $y = 10m$, and $21 \mid (10m+1)$. Now, let $m = 21k+2$. Then $10m+1 = 210k+21$. So $21 \mid (10m+1)$. So $(14, 10 \cdot (21k+2))$ is an eligible pair, $k = 0, 1, 2, \ldots$. We have found infinitely many pairs.

To find the smallest pair, note that by taking $k = 0$ we have $(14, 20)$ as a pair, so any smaller pair (x, y) will have $x + y \leq 34$. For $x = 14$, one can check this is the smallest by ruling out $y = 15, 16, 17, 18, 19$ with brute force. If $x > 14$, the next eligible one is $x = 20$. So $y > 20$. So $x + y > 40 > 34$. Therefore, $(14, 20)$ is the smallest pair.

Alternate Solution to v#3

v#3 Because $x(x+1)$ divides $y(y+1)$ then x and $(x+1)$ can be factored as $x = AB$ and $(x+1) = CD$ where AC divides y and BD divides $(y+1)$. What values of A, B, C, D are possible? We now show that none of A, B, C, or D share a factor (they are relatively prime), and all are greater than one.

If $A = 1$ then x divides $(y+1)$, a contradiction, so A is not 1. Similarly, B, C, and D are not 1. Also, if x and $(x+1)$ have a common factor p then p is a factor of $(x+1) - x = 1$, impossible. So x and $(x+1)$ have no common factors, and similarly y and $(y+1)$ have no common factors. Also, A and B (or C and D) have no common factors, because if they did then y and $(y+1)$ would have a common factor.

With this in mind, the smallest values possible for factors of A, B, C, D are then 2,3,5,7. The smallest way to get $x, x+1$ from these is $x = 2*7 = 14$, $x+1 = 3*5 = 15$ (another is $x = 20$, $x+1 = 21$). It remains to find y and $y+1$.

Since $AC = 10$ divides y then $y = 10m$, and likewise $y+1 = 21m'$, for some m and m'. But then $21m' = 10m+1$. The simplest solution is $m' = 1$, $m = 2$. More generally, any $m' = 10*k+1$ and $m = 21*k+2$ will work. This gives infinitely many solutions.

Among these, the smallest is $y = 20$, $y+1 = 21$, for a total $x+y = 14+20 = 34$. Any smaller total $x+y < 34$ would have $x < 17$ or $y < 17$. But $x(x+1) \leq y(y+1)$, so $x \leq y$ and at least $x < 17$. By checking all values we find that for $x < 17$ the only value of x, $(x+1)$ where x and $(x+1)$ each have at least two different prime factors is $x = 14$, so 34 is the smallest total.

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v#4. The tight integer problem: If n and m are tight, show that nm is tight.

Take any $2nm - 1$ integers. Taking the first group of $2n - 1$, we can find exactly n such that their sum s_1 is divisible by n. Set these aside, and take another group of $2n - 1$ to obtain another n whose sum s_2 is divisible by n. Continuing in this manner, one obtains $2m - 1$ disjoint collections of n of the original integers with sums $s_1, \ldots, s_{2m-1}$, each divisible by n. Set $s_j = k_j n$. Note that there are $2m - 1$ of the k_j. Since m is also tight, one can pick exactly m of the k_j whose sum is divisible by m. Call these $k_1^*, \ldots, k_m^*$ and write $s_j^* = k_j^* n$. The union of the corresponding collections contains exactly nm elements. Since $s_1^* + \cdots + s_m^* = (k_1^* + \cdots + k_m^*)n$ and the sum $k_1^* + \cdots + k_m^*$ is divisible by m, we are done.

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v#5. Let $d_1 \le d_2 \le d_3$ be the digits of a (not necessarily in that order). Then $b = d_1d_2d_3$ and $c = d_3d_2d_1$. Clearly $d_3 > d_1$, or we'll have $c - b = 0$. Now, the middle digit for both b, c are the same. We conclude that $c - b = a$ must have 9 as its middle digit. Also, $c - b = (d_3 - d_1) \cdot 100 + d_1 - d_3 = 99 \cdot (d_3 - d_1)$. So $9 \mid a$. This means 9 divides the sum of the digits of a, and thus 9 divides the sum of the two other digits of a. Now, there are only nine possibilities:

$$198, \quad 297, \quad 396, \quad 495, \quad 594, \quad 693, \quad 792, \quad 891, \quad 990.$$

It is a brute force check that $a = 495$.

Alternate Solution to v#5

v#5. Let $d_1 \le d_2 \le d_3$ be the 3 digits of a. So we have $c = d_3d_2d_1$ and $b = d_1d_2d_3$. From the fact that b and c have the same middle digit and $d_3 > d_1$ (if not all d_i would be equal and so $a = 000$) it follows that the middle digit of a is 9, so that $d_3 = 9$. It follows that a is either d_19d_2 or d_29d_1. The second case is impossible ($10 + d_1 - 9 = d_1$ would follow from the last row of $c - b = a$) so we have $2d_1 = 8$ and $d_1 = d_2 - 1$. From which $a = 495$.