4 Group

1. Suppose $1 \le a < b < c < d \le 100$ are four natural numbers. What is the minimum possible value for $\frac{a}{b} + \frac{c}{d}$?

> **Solution:** $\frac{21}{100}$. We should assume $a = 1, c = b + 1, d = 100$, so the sum is $\frac{1}{b} + \frac{b+1}{100}$. But $\frac{1}{b} + \frac{b}{100} \ge 2\sqrt{\frac{1}{100}} = \frac{1}{5}$ and equality can hold when $b = 10$.

2. Let ABC be a triangle with integer-length sides so that $\angle ABC = 120°$ and $AB = 14$. What is the largest possible value of AC?

> **Solution:** 74. Draw the perpendicular from A to meet the extension of BC at D, so $AD = 7\sqrt{3}$ and $BD = 7$ as $\angle ABD = 60°$. The fact that BC is an integer means that $CD = BC + 7$ is also an integer, so we want the largest integers BC and CD so that $AC^2 = AD^2 + CD^2 = 147 + CD^2$. As 147 is odd, this is maximized when $AC = CD + 1$. Solving gives $AC = 74$ and $BC = CD - 7 = 66$.

3. We say a sequence of natural numbers $(a_1, \ldots, a_k)$ is "cyclic" if the last digit of a_i is the first digit of a_{i+1} for each $i = 1, 2, \ldots, k - 1$, and the last digit of a_k is the first digit of a_1. For instance, $(8118), (23, 32)$, and $(12, 24, 48, 801)$ are cyclic. At least how many integers must be removed from the set $100, 101, \ldots, 999$ so no cyclic sequences can be formed using the remaining integers?

> **Solution:** 450. First of all we have to take away integers of the form $\overline{aba}$. There are 90 of them. Then whenever we want to keep an integer $\overline{abc}$ with $a \ne c$ and $c \ne 0$, we have to take away all integers of the form $\overline{cxa}$; so for every $a \ne c$ with $a, c \ne 0$, we have to take away at least 10 integers with the first and the last digit being a, c, giving a total of 360. Taking away 450 integers is possible: take away all integers $\overline{abc}$ with $a \le c$.

4. In this problem each of the letters A, C, E, H, M, S, T represents a different digit from 0 to 9, and M, T, H are not zero. So that each of $MATH$, $TECH$, and $HSMC$ represent 4-digit natural numbers. Given the requirement that $MATH + TECH = HSMC$, what is the least possible value of $HSMC$?

Solution: 4618. Since $M \neq T$ are both non-zero, $H \geq M + T \geq 3$. If $H = 3$, then $C = H + H = 6$, but then $6 + T \equiv M \pmod{10}$ has no solution subjects to $M + T \leq H = 3$. If $H = 4$, then $C = 8$, and $8 + T = M \pmod{10}$ has a solution consistent with $M + T \leq H = 4$: $T = 3, M = 1$. Now the digits $0, 2, 5, 6, 7, 9$ are available for A, E, S and the only requirement is that $A+E+1 = S$. The least possible value of S is $S = 6$ with $A, E = 0, 5$ in some order. This gives $MATH = 1034$ (or 1534), $TECH = 3584$ (or 3084), and $HSMC = 4618$.

5. Using only $+, -, \times, \div$, and parentheses, form the following lists of integers into arithmetic expressions equal to 24: (i) 2,5,10,13; (ii) 5,7,10,11; (iii) 1,7,13,13; (iv) 3,4,8,13. Every integer in a list must be used the same number of times as it appears in the list.

Solution: (i) $2\times13-10\div5 = 24$; (ii) $(10-5)\times7-11 = 24$; (iii) $(13\times13-1)\div7 = 24$; (iv) $8 \div (13 \div 3 - 4) = 24$.

6. Find a permutation $a_1, a_2, \ldots, a_{16}$ of $1, 2, \ldots, 16$ such that whenever $i < j < k$, a_i, a_j, a_k do not form an arithmetic sequence, i.e. $a_k - a_j \neq a_j - a_i$.

Solution: A possible construction method is to notice that if $b_1, \ldots, b_m$ is a permutation of $1, \ldots, m$ satisfying the required property, then $2b_1, \ldots, 2b_m, 2b_1-1, \ldots, 2b_m-1$ is a permutation of $1, 2 \ldots, 2m$ that has the same property: the first and the last term of a 3-term arithmetic sequence have the same parity, so any 3-term arithmetic sequence in the new permutation must be completely from the first half or the second half, which is impossible by induction. Starting with $m = 1$, we can inductively construct a desired permutation.

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$\Rightarrow 2, 1$
$\Rightarrow 4, 2, 3, 1$
$\Rightarrow 8, 4, 6, 2, 7, 3, 5, 1$
$\Rightarrow 16, 8, 12, 4, 14, 6, 10, 2, 15, 7, 11, 3, 13, 5, 9, 1.$

7. A Fermi question is a question where you need to make educated guesses in order to estimate the order of magnitude of a certain quantity. What follows is a Fermi question. How many times is the mass of Earth bigger than that of an adult Asian elephant? Give your answer as the multiplicatively closest power of 10, i.e., if your answer is N, answer 10^n where n is the integer closest to $\log_{10} N$. A 10 times magnitude difference will be accepted as correct answer.

Solution: 10^{21}. The radius of Earth is around 6000 km. One way to make educated guess is to start with two cities that you know the estimated distance between them as well as their positions on the map, use these data to estimate the circumference of Earth. So Earth has volume of around 10^{12} km^3; with an average density of around 5 g/cm^3. (Earth should be denser than water, whose density is 1 g/cm^3, but not 10 times denser.) Earth's mass is around 5×10^{24} kg. An adult Asian elephant weights around 5000 kg. (Compare the measure of an elephant and a human, and use the fact that human and elephant have similar composition.)

8. How many candies are there in the jar?