2 Ciphering

1. How many three digit square numbers end in 9?

> **Solution:** 4. The squares of numbers ending in 3 or 7 end in 9. The squares of numbers between 10 and 31 have three digits. So $13^2, 17^2, 23^2$, and 27^2 end in 9.

2. Let $A_1, A_2, \ldots, A_8$ be the 8 vertices, in cyclic order, of a regular octagon. Find the angle $\angle A_1A_3A_7$, answer in degrees.

> **Solution:** $45°$. Note the octogon is inscribed in a circle $\odot O$ and arc A_1A_7 is one-fourth of it, so $\angle A_1A_3A_7 = \frac{1}{2}\angle A_1OA_7 = \frac{1}{2}(360° \times \frac{1}{4})$, where O is the center of the octogon.

3. Let $f(x) = 3x^4 + 2x^2 - x + 2$ and $g(x) = 7x^6 + 2x^4 + 2x - 4$. What is the sum of the coefficients of the polynomial $f(x) \cdot g(x)$?

> **Solution:** 42. The sum of the coefficients of a polynomial $h(x)$ is $h(1)$, so the answer is $f(1) \cdot g(1) = 6 \cdot 7$.

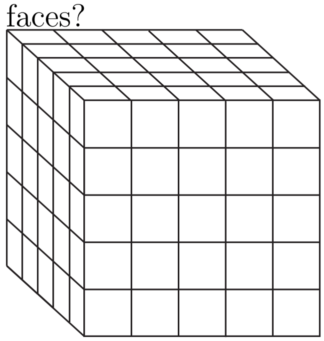
4. Let N be the number of ways to arrange 17 gold beads and 17 white beads in a sequence. What is the largest prime factor of N?

> **Solution:** 31. $N = \binom{34}{17} = \frac{34!}{17! \times 17!}$. 31 is the largest prime dividing 34! while 31 does not divide the denominator, so the answer is 31.

5. Write a date in the form MMDDYYYY; for example, today is 03122016. What is the next date in this form with all 8 digits distinct?

> **Solution:** 06172345. Note that every month contains a digit of 0 or 1, and every day contains a digit of 0, 1, or 2. We assume our year starts with 2 first, so MMDD contains both the digits 0 and 1, and the earliest possible year is 2345. The earliest month is 06 (01 is not possible by considering day), and the earliest day is 17.

6. Consider a $5 \times 5 \times 5$ cube with the outside surface painted blue. Buzz cuts the cube into 5^3 unit cubes, then picks a cube at random. Given that the cube Buzz picked has at least one painted blue face, what is the probability that the cube has exactly two blue faces?



Solution: $\frac{18}{49}$. There are $5^3 - 3^3 = 98$ cubes with at least one face painted blue. There are $12 \times 3 = 36$ cubes with two faces painted blue. So the probablity is $\frac{36}{98} = \frac{18}{49}$.

7. How many ways can one choose a non-empty set of integers whose sum is zero from the set $\{-3, -2, -1, 0, 1, 2, 3, 6\}$?

Solution: 21. We first do the same problem without the 0 and the non-empty condition: if one chooses 6 then the only way to get a zero sum is to choose $-1, -2, -3$ also; if one does not choose 6 and tries to choose odd number of integers then it must be $\{-3, 1, 2\}$ or $\{-2, -1, 3\}$, otherwise one can pick any union of the sets $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}$. So we have $1 + 2 + 2^3 = 11$ subsets here, and the answer of original question is $2 \times 11 - 1$ as every valid non-empty subset can be chosen to further include 0 or not.

8. Suppose $\tan x + \cot x + \sec x + \csc x = 6$. Find the value of $\sin x + \cos x$.

Solution: $\frac{4}{3}$. Write $t := \sin x + \cos x$, we have $6 = \tan x + \cot x + \sec x + \csc x = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} + \frac{\sin x + \cos x}{\sin x \cos x} = \frac{2}{t^2 - 1} + \frac{2t}{t^2 - 1} = \frac{2}{t-1}$, so $t = \frac{4}{3}$.

9. What is the period length of the repeating sequence of digits in the duodecimal (base 12) expression for $\frac{1}{2016}$? Here 2016 is in decimal, and your answer should be a decimal number.

Solution: 6. The length is the smallest positive integer n such that $\frac{1}{2016} = \frac{A}{12^r(12^n-1)}$ for some integers A, r. We know $2016 = 2^5 \times 3^2 \times 7$, 2 and 3 are factors of 12 and 7 is a prime which is coprime with 12; while 6 is the smallest n satisfying $12^n \equiv 1 \pmod 7$.

10. The three altitudes of a triangle have lengths $\frac{2}{9}$, $\frac{1}{5}$, and $\frac{2}{17}$. What is the area of this triangle?

Solution: $\frac{1}{36}$. Let a, b, and c be the lengths of the sides corresponding to the altitudes of lengths $\frac{2}{9}$, $\frac{1}{5}$, and $\frac{2}{17}$ respectively. Let $K > 0$ be the area of the triangle. We have $K = \frac{1}{9}a = \frac{1}{10}b = \frac{1}{17}c$, hence $a = 9K$, $b = 10K$, and $c = 17K$. Then, we can utilize Heron's Formula. The semiperimeter of the triangle is $s = \frac{1}{2}(a+b+c) = 18K$, and the area of the triangle is $K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{18K \cdot K \cdot 8K \cdot 9K} = 36K^2$. Solving gives $K = \frac{1}{36}$.