Georgia Tech High School Math Competition

Ciphering Test—Solutions

February 15, 2014

1. Ivan is in a class of 16 students. Ivan got 45 on the test while the class average was 60, but he decides to appeal the grade. How many additional points does he need in order for his score to equal the new class average, assuming no one else's grade changes?

> **Solution:** 16. Suppose the number of points Ivan needs is x. The class total after the grade change is $16 \cdot 60 + x$, so x must satisfy $45 + x = (16 \cdot 60 + x)/16$.

2. Riesling tosses a ring of radius 1 at a square of side length 6 on the ground. She wins if the ring lands entirely within the square. If the center of the ring is uniformly distributed within the square, what's the probability that Riesling wins?

> **Solution:** 4/9. For the ring to land entirely within the square, the center must land no closer than distance 1 from any edge: this "good" region is a square of side length 4. Since the good region has area 16 and there is a total area of 36, the probability is $16/36 = 4/9$.

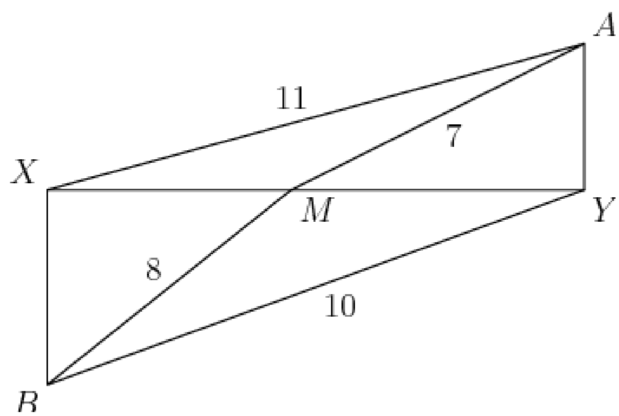
3. Let r be a real number with $r^3 + 1/r^3 = 18$. Find $r + 1/r$.

> **Solution:** 3. First, observe that
>
> $$(r + 1/r)^3 = r^3 + 3r + 3/r + 1/r^3 = 18 + 3(r + 1/r).$$
>
> Set $x = r + 1/r$. Then $x^3 - 3x - 18 = 0$, which has unique real solutions $x = 3$.

4. What is the smallest integer $n > 1$ such that there exists integers $a_1, a_2, \ldots, a_n$ satisfying $a_1 + a_2 + \ldots + a_n = a_1 a_2 \ldots a_n = 2014$?

Solution: 5. We can achieve $n = 5$ with $a_1 = 2014, a_2 = a_4 = 1, a_3 = a_5 = -1$. To show smaller n doesn't work, WLOG a_1 is the largest number, the facts that $a_1 \geq 2014/n$ and a_1 is a factor of 2014 force a_1 to be either 2014 or 1007, reduce the problem to $a_2 \ldots a_n = 1$ or 2, and $a_2 + \ldots + a_n = 0$ or 1007, it is not hard to see those cases are impossible.

5. Let XY be a line segment and let M be a point on it. Let A, B be points such that AY, BX are perpendicular to XY. If $AX = 11, AM = 7, BY = 10, BM = 8$. Find the value of $(XM)^2 - (YM)^2$.



Solution: 36. Denote by x, y the length of XM, YM respectively. We have $(x + y)^2 + AY^2 = 11^2, y^2 + AY^2 = 7^2$, subtracting the second equation from the first gives $x^2 + 2xy = 72$, similar consideration from B gives $y^2 + 2xy = 36$, so $x^2 - y^2 = 36$.

6. What is the remainder when $1^{2014} + 2^{2014} + 3^{2014} + \cdots + 106^{2014}$ is divided by 107?

Solution: 106. Since 107 is prime, Fermat's little theorem tells us that $a^{106} \equiv 1 \pmod{107}$ for all $a \in \{1, ..., 106\}$. Since 106 divides 2014, this implies that $a^{2014} \equiv 1 \pmod{107}$ as well. Summing up all 106 terms produces a remainder of 106.

7. Find the value of

$$\frac{\ln \tan 1^\circ + \ln \tan 2^\circ + \ldots + \ln \tan 45^\circ}{\ln \tan 46^\circ + \ln \tan 47^\circ + \ldots + \ln \tan 89^\circ}.$$

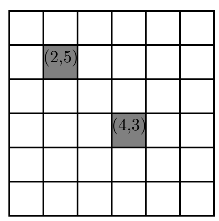
(ln denotes the natural logarithm, which is $\log_e$.)

Solution: -1. We have $\ln\tan 45° = 0$, for other term, $\ln\tan\theta + \ln\tan(\frac{\pi}{2} - \theta) = \ln(\tan\theta\tan(\frac{\pi}{2} - \theta)) = \ln 1 = 0$, so $\ln\tan\theta = -\ln\tan(\frac{\pi}{2} - \theta)$. Hence the top sum is the negative of the bottom sum.

8. Find the sum of all real solutions x to the equation $(x^2 - 9x + 19)^{(x^3-3x^2+2x)} = 1$.

Solution: 21. If a and b are real numbers, then $a^b = 1$ if either $a = 1$, or $b = 0, a \neq 0$, or $a = -1$ and b is an even integer. So, we have the following cases: Case 1: $x^2 - 9x + 19 = 1$: We have $0 = x^2 - 9x + 18 = (x - 3)(x - 6)$, and so, $x = 3, 6$. Case 2: $x^3 - 3x^2 + 2x = 0$: We have $0 = x^3 - 3x^2 + 2x = x(x - 1)(x - 2)$, and so $x = 0, 1, 2$, all three of which satisfy $x^2 - 9x + 19 \neq 0$. Case 3: $x^2 - 9x + 19 = -1$: We have $0 = x^2 - 9x + 20 = (x - 4)(x - 5)$, and so $x = 4, 5$, both of which make $x^3 - 3x^2 + 2x$ even. Therefore, the solutions are $x = 0, 1, 2, 3, 4, 5, 6$.

9. How many rectangles are there in a 6×6 chessboard that contain neither the cell $(2, 5)$ nor $(4, 3)$? (The rows and columns are indexed by $1, \ldots, 6$.)



Solution: 233. To count all rectangles on the board, there are $1+2+3+4+5+6 = 21$ choices for the left and right bounds. This can be obtained by counting the number of choices for each width, or by noting that these correspond to the number pairs of vertical cell boundaries, which is $\binom{7}{2}$. The same count for top and bottom gives us $21^2 = 441$ total rectangles. To count rectangles containing $(2, 5)$, the left side must be before 2, so there are 2 choices, while the right must be after 2 giving 5 choices, which gives 10 in total. Top and bottom also has 10 choices, so there are 100 rectangles containing $(2, 5)$. Similarly we count rectangles containing $(4, 3)$ which is $12^2 = 144$ and the number containing both which is $6^2 = 36$. By inclusion-exclusion, the answer is $441 - 100 - 144 + 36 = 233$.

10. Find the last six digits of 3^{2000}.

Solution: 440001.

$$3^{2000} = (80+1)^{500} = 1 + \binom{500}{1}80 + \binom{500}{2}80^2 + \binom{500}{3}80^3 + \ldots + 80^{500}.$$

It is not hard to get a 10^6 factor for each term after the third term (the 80^k term takes over since the seventh term, and before that get some extra zeros from the 500 factor in $\binom{500}{k}$), so the answer is just the last six digits of the sum of first three terms, i.e. $1 + 40000 + 400000$.