HSMC 2017 Proof

1. Show that every non-negative integer can be written in the form $x^2 + y^2 - z^2$, where $x, y, z$ are some non-negative integers.

**Solution:** Suppose $n$ is odd, say $n = 2m + 1$. Then $n = 0^2 + (m + 1)^2 - m^2$. Say $n$ is even, say $n = 2m + 2$ (note that $0 = 0^2 + 0^2 - 0^2$), then $n = 1^2 + (m + 1)^2 - m^2$.

2. Find, with proof, the smallest possible sum of six distinct prime numbers which form an arithmetic progression.

**Solution:** 492.

Let $p$ be the smallest of these primes and $d \geq 1$ be the common difference. So the six primes are $p_k = p + kd$ for $k = 0, 1, \ldots, 5$, and their sum is $p + (p + d) + (p + 2d) + \cdots + (p + 5d) = 6p + 15d$.

If $p = 2$, then $p_2 = 2 + 2d = 2(1 + d)$, which is not prime. If $p = 3$, then $p_3 = 3 + 3d = 3(1 + d)$, which is not prime. If $p = 5$, then $p_5 = 5 + 5d = 5(1 + d)$, which is not prime. Therefore, $p \geq 7$.

Since $p \geq 7$, none of the primes $p_0, p_1, \ldots, p_5$ can be divisible by 5. So by the pigeonhole principle, two of $\{p_0, p_1, p_2, p_3, p_4\}$ are congruent mod 5 (lets say $p_i \equiv p_j \pmod{5}$ and $0 \leq i < j \leq 4$). Then, $p_j - p_i = (j - i)d \equiv 0 \pmod{5}$. Since $j - i$ is between 1 and 4, we get that $d \equiv 0 \pmod{5}$. Similarly, we get that $d \equiv 0 \pmod{2}$ and $d \equiv 0 \pmod{3}$. Therefore $d$ must be divisible by 2, 3, and 5, and so $d \geq 30$.

These two restrictions give $6p + 15d \geq 6 \cdot 7 + 15 \cdot 30 = 492$. Trying $p = 7$, $d = 30$ gives 7, 37, 67, 97, 127, 157 as the numbers in the sequence, which are all prime. This gives a sum of 492. Thus, 492 is the minimum sum.
3. Suppose $\triangle ABC$ is a triangle with $\angle ABC = 90^\circ$. Let $D$ be the midpoint of $AC$ and $E$ be the midpoint of $AD$. Suppose we have $BD$ is the internal bisector of $\angle CBE$. Find, with proof, the value of $\angle BCA$.

**Solution:** $30^\circ$.

Let $F$ be the midpoint of $BC$. We see that $\triangle BDF$ and $\triangle CDF$ are congruent (SSS). Thus we see that $\angle DFB = 90^\circ$. Since $BD$ is the internal bisector of $\angle CBE$, by the angle bisector theorem we have

\[
\frac{BE}{BC} = \frac{ED}{DC} = \frac{1}{2},
\]

i.e. $BE = \frac{1}{2}BC = BF$. Hence we have $\triangle BDE$ and $\triangle BDF$ are congruent (SAS). So $\angle DEB = 90^\circ$, and thus $\angle BCA = \frac{1}{3}90^\circ = 30^\circ$.

**Alternate solution:** Let $\angle CBD = \alpha$ and $\angle ABE = \beta$. By applying Sine rule to $\triangle ABE$ and $\triangle BDE$ we get

\[
\frac{\sin(\alpha + \beta)}{\sin(\beta)} = \frac{BE}{AE} = \frac{BE}{ED} = \frac{\sin(2\alpha)}{\sin(\alpha)}
\]

Simplifying, we have

\[
\sin(\alpha + \beta) = 2\cos(\alpha)\sin(\beta) = \sin(\alpha + \beta) - \sin(\alpha - \beta)
\]

So $\sin(\alpha - \beta) = 0$, and hence $\alpha = \beta = \angle BCA = 30^\circ$. 

---

Page 2
4. Find, with proof, the number of positive integer solutions to the equation \( a + b + c = n \) (where \( n \) is a positive integer), so that \( a, b \) and \( c \) are lengths of sides of a non-degenerate triangle.

**Solution:** \( \left( \frac{n}{2} - 1 \right) \) if \( n \) even, \( \left( \frac{n+1}{2} \right) \) if \( n \) odd.

First of all we will use a transformation to rewrite the triangle inequality in simpler terms. If we set \( x = (b + c - a)/2, y = (c + a - b)/2, z = (a + b - c)/2 \) (equivalently, \( a = y + z, b = z + x, c = x + y \)), then we see that the triangle inequalities simply translates to saying \( x, y, z \) are positive. (Geometrically, \( x, y, z \) are lengths of segments cut off by the incircle).

We see that \( 2x, 2y, 2z \) must always be positive integers, and in fact must all have the same parity (for example \( 2x + 2y = 2c \)). Thus, \( 2x, 2y, 2z \) are all even (resp. odd) iff \( n \) is even (resp. odd). We will look at these two cases separately.

Case 1: If \( n \) is even, then we are looking for:
- the number of positive integer solutions to \( x + y + z = \frac{n}{2} \)
- the number of non-negative integer solutions to \( x' + y' + z' = \frac{n}{2} - 3 \) (where \( x' = x - 1, y' = y - 1 \) and \( z' = z - 1 \))
- \( \left( \frac{n}{2} - 1 \right) \) (by stars and bars).

Case 2: If \( n \) is odd, then we are looking for:
- the number of positive half-integer solutions to \( x + y + z = \frac{n}{2} \)
- the number of non-negative integer solutions to \( x' + y' + z' = \frac{n-3}{2} \) (where \( x' = x - \frac{1}{2}, y' = y - \frac{1}{2} \) and \( z' = z - \frac{1}{2} \))
- \( \left( \frac{n+1}{2} \right) \) (by stars and bars).
5. Let \( w, x, y \) and \( z \) be positive real numbers satisfying \((1 + w)(1 + x)(1 + y)(1 + z) = 4\). Prove that

\[ wxyz + \frac{1}{wxyz} \geq 34. \]

**Solution:** For any two positive reals \( a \) and \( b \), we have

\[(1 + a)(1 + b) = 1 + a + b + ab \geq 1 + 2\sqrt{ab} + ab = (1 + \sqrt{ab})^2.\]

By applying the above inequality thrice to the pairs \((w, x)\), \((y, z)\) and \((\sqrt{wx}, \sqrt{yz})\) we get:

\[(1 + w)(1 + x)(1 + y)(1 + z) \geq (1 + \sqrt{wx})^2(1 + \sqrt{yz})^2 \geq (1 + \sqrt{wxyz})^4.\]

This can also be obtained by applying AM-GM inequality to homogeneous parts of \((1 + w)(1 + x)(1 + y)(1 + z)\).

Let \( u = \sqrt{wxyz} \). Then we have

\[ 4 \geq (1 + u)^4 \iff 2 \geq (1 + u)^2 \iff 1 \geq 2u + u^2 \iff \frac{1}{u} - u \geq 2 \]

\[ \iff (\frac{1}{u} - u)^2 \geq 4 \iff \frac{1}{u^2} + u^2 \geq 6 \]

\[ \iff (\frac{1}{u^2} + u^2)^2 \geq 36 \iff \frac{1}{u^4} + u^4 \geq 34. \]
6. (Tiebreaker) A king can move one-step horizontally, vertically or diagonally. Is there a path on the $4 \times 4$ chessboard passing every square exactly once so that the king starts with a diagonal move and then alternates between horizontal-vertical and diagonal moves (a) starting from bottom-left corner and ending at bottom-right corner? (b) starting from bottom-left corner and ending at top-right corner?

Solution: (a) There is such an path from bottom-left square to bottom-right square as shown.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>6</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>15</td>
<td>13</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>14</td>
<td>16</td>
</tr>
</tbody>
</table>

(b) If there is such a tour, the king would have to make exactly 8 diagonal moves and 7 horizontal-vertical moves. Only the horizontal-vertical moves change the color of the square the king is in. Since the bottom-left square and top-right squares have the same color, there is no such path.

Problems contributed by Jose Acevedo, Marcel Celaya, Santhosh Karnik, George Kerchev, Sudipta Kolay, Peter Ralli, Libby Taylor, Hagop Tossounian and Chi Ho Yuen.