HSMC 2017 Proof

1. Show that every non-negative integer can be written in the form $x^2 + y^2 - z^2$, where x, y, z are some non-negative integers.

> **Solution:** Suppose n is odd, say $n = 2m + 1$. Then $n = 0^2 + (m + 1)^2 - m^2$. Say n is even, say $n = 2m + 2$ (note that $0 = 0^2 + 0^2 - 0^2$), then $n = 1^2 + (m + 1)^2 - m^2$.

2. Find, with proof, the smallest possible sum of six distinct prime numbers which form an arithmetic progression.

> **Solution:** 492.
> Let p be the smallest of these primes and $d \geq 1$ be the common difference. So the six primes are $p_k = p + kd$ for $k = 0, 1, \ldots, 5$, and their sum is $p + (p + d) + (p + 2d) + \cdots + (p + 5d) = 6p + 15d$
>
> If $p = 2$, then $p_2 = 2 + 2d = 2(1 + d)$, which is not prime. If $p = 3$, then $p_3 = 3 + 3d = 3(1 + d)$, which is not prime. If $p = 5$, then $p_5 = 5 + 5d = 5(1 + d)$, which is not prime. Therefore, $p \geq 7$.
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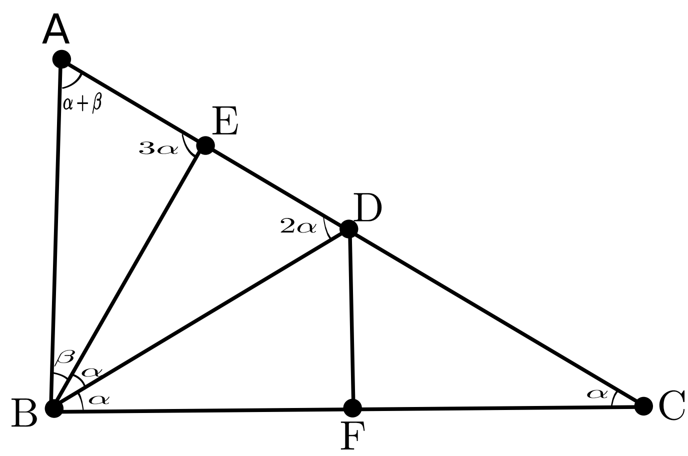
> Since $p \geq 7$, none of the primes $p_0, p_1, \ldots, p_5$ can be divisible by 5. So by the pigeonhole principle, two of $\{p_0, p_1, p_2, p_3, p_4\}$ are congruent mod 5 (lets say $p_i \equiv p_j \pmod 5$ and $0 \leq i < j \leq 4$). Then, $p_j - p_i = (j - i)d \equiv 0 \pmod 5$. Since $j - i$ is between 1 and 4, we get that $d \equiv 0 \pmod 5$. Similarly, we get that $d \equiv 0 \pmod 2$ and $d \equiv 0 \pmod 3$. Therefore d must be divisible by 2, 3, and 5, and so $d \geq 30$.
>
> These two restrictions give $6p + 15d \geq 6 \cdot 7 + 15 \cdot 30 = 492$. Trying $p = 7$, $d = 30$ gives $7, 37, 67, 97, 127, 157$ as the numbers in the sequence, which are all prime. This gives a sum of 492. Thus, $\boxed{492}$ is the minimum sum.

3. Suppose ΔABC is a triangle with $\angle ABC = 90°$. Let D be the midpoint of AC and E be the midpoint of AD. Suppose we have BD is the internal bisector of $\angle CBE$. Find, with proof, the value of $\angle BCA$.



Solution: 30°.
Let F be the midpoint of BC. We see that ΔBDF and ΔCDF are congruent (SSS). Thus we see that $\angle DFB = 90°$. Since BD is the internal bisector of $\angle CBE$, by the angle bisector theorem we have

$$\frac{BE}{BC} = \frac{ED}{DC} = \frac{1}{2},$$

i.e. $BE = \frac{1}{2}BC = BF$. Hence we have ΔBDE and ΔBDF are congruent (SAS). So $\angle DEB = 90°$, and thus $\angle BCA = \frac{1}{3}90° = 30°$.

Alternate solution: Let $\angle CBD = \alpha$ and $\angle ABE = \beta$. By applying Sine rule to ΔABE and ΔBED we get

$$\frac{\sin(\alpha+\beta)}{\sin(\beta)} = \frac{BE}{AE} = \frac{BE}{ED} = \frac{\sin(2\alpha)}{\sin(\alpha)}$$

Simplifying, we have

$$\sin(\alpha+\beta) = 2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

So $\sin(\alpha-\beta) = 0$, and hence $\alpha = \beta = \angle BCA = 30°$.

4. Find, with proof, the number of positive integer solutions to the equation $a+b+c=n$ (where n is a positive integer), so that a, b and c are lengths of sides of a non-degenerate triangle.

Solution: $\binom{\frac{n}{2}-1}{2}$ if n even, $\binom{\frac{n+1}{2}}{2}$ if n odd.
First of all we will use a transformation to rewrite the triangle inequality in simpler terms. If we set $x=(b+c-a)/2, y=(c+a-b)/2, z=(a+b-c)/2$ (equivalently, $a=y+z, b=z+x, c=x+y$), then we see that the triangle inequalities simply translates to saying x, y, z are positive. (Geometrically, x, y, z are lengths of segments cut off by the incircle).
We see that $2x, 2y, 2z$ must always be positive integers, and infact must all have the same parity (for example $2x+2y=2c$). Thus, $2x, 2y, 2z$ are all even (resp. odd) iff n is even (resp. odd). We will look at these two cases separately.
Case 1: If n is even, then we are looking for:
the number of positive integer solutions to $x+y+z=\frac{n}{2}$
= the number of non-negative integer solutions to $x'+y'+z'=\frac{n}{2}-3$ (where $x'=x-1$, $y'=y-1$ and $z'=z-1$)
$=\binom{\frac{n}{2}-1}{2}$ (by stars and bars).
Case 2: If n is odd, then we are looking for:
the number of positive half-integer solutions to $x+y+z=\frac{n}{2}$
= the number of non-negative integer solutions to $x'+y'+z'=\frac{n-3}{2}$ (where $x'=x-\frac{1}{2}$, $y'=y-\frac{1}{2}$ and $z'=z-\frac{1}{2}$)
$=\binom{\frac{n+1}{2}}{2}$ (by stars and bars).

5. Let w, x, y and z be positive real numbers satisfying $(1+w)(1+x)(1+y)(1+z) = 4$. Prove that

$$wxyz + \frac{1}{wxyz} \geq 34.$$

Solution: For any two positive reals a and b, we have

$$(1+a)(1+b) = 1 + a + b + ab \geq 1 + 2\sqrt{ab} + ab = (1+\sqrt{ab})^2.$$

By applying the above inequality thrice to the pairs (w, x), (y, z) and $(\sqrt{wx}, \sqrt{yz})$ we get:

$$(1+w)(1+x)(1+y)(1+z) \geq (1+\sqrt{wx})^2(1+\sqrt{yz})^2 \geq (1+\sqrt[4]{wxyz})^4.$$

This can also be obtained by applying AM-GM inequality to homogeneous parts of $(1+w)(1+x)(1+y)(1+z)$.

Let $u = \sqrt[4]{wxyz}$. Then we have

$$\begin{aligned} 4 \geq (1+u)^4 \iff 2 \geq (1+u)^2 \iff 1 \geq 2u + u^2 \iff & \frac{1}{u} - u \geq 2 \\ \iff & (\frac{1}{u} - u)^2 \geq 4 \iff \frac{1}{u^2} + u^2 \geq 6 \\ \iff & (\frac{1}{u^2} + u^2)^2 \geq 36 \iff \frac{1}{u^4} + u^4 \geq 34. \end{aligned}$$

6. (Tiebreaker) A king can move one-step horizontally, vertically or diagonally. Is there a path on the 4×4 chessboard passing every square exactly once so that the king starts with a diagonal move and then alternates between horizontal-vertical and diagonal moves
(a) starting from bottom-left corner and ending at bottom-right corner?
(b) starting from bottom-left corner and ending at top-right corner?

Solution: (a) There is such an path from bottom-left square to bottom-right square as shown.

7	6	11	10
5	8	9	12
4	2	15	13
1	3	14	16

(b) If there is such a tour, the king would have to make exactly 8 diagonal moves and 7 horizontal-vertical moves. Only the horizontal-vertical moves change the color of the square the king is in. Since the bottom-left square and top-right squares have the same color, there is no such path.

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