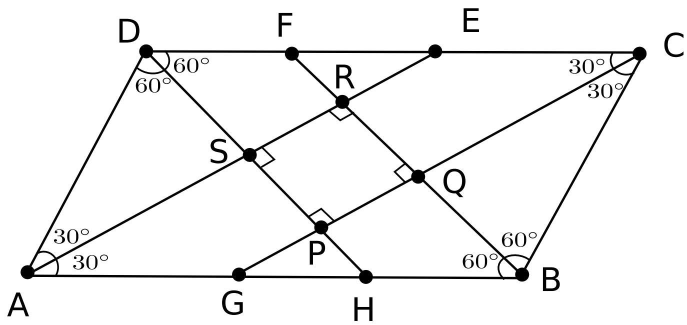
HSMC 2017 Group

1. Suppose $ABCD$ is a parallelogram with side lengths $AB = 3$, $AD = 2$. and $\angle DAB = 60^\circ$. Find the area of the parallelogram formed by the intersections of the internal bisectors of of $\angle DAB$, $\angle ABC$, $\angle BCD$, $\angle CDA$.



Solution:

Let E, F be points on CD, and G, H be points on AB such that AE, BF, CG and GH are the internal bisectors, as shown. Suppose they intersect in the points P, Q, R and S as shown.

By angle chasing we see that $PQRS$ is a rectangle. We observe that $AH = BG = DE = CF = 2 = AD = BC$, and $AG = BH = CE = DF = 1$.

By similarity, $\frac{SP}{SH} = \frac{AG}{AH} = \frac{1}{2}$, and thus $SP = \frac{1}{2}SH = \frac{1}{2}AH \sin 30^\circ = \frac{1}{2}$.

By similarity, $\frac{PQ}{GQ} = \frac{HB}{GB} = \frac{1}{2}$, and thus $PQ = \frac{1}{2}GQ = \frac{1}{2}GB \sin 60^\circ = \frac{\sqrt{3}}{2}$.

Consequently, area of parallelogram $PQRS$ is $\frac{\sqrt{3}}{4}$.

2. Inspector Lestrade visited an island to solve a case of a missing jewel. The inhabitants of the islands were either knights, who always spoke the truth and never stole anything, or knaves, who always lied and sometimes stole things. Inspector Lestrade knew that out of the six suspects exactly one had stolen the jewel. Here is what they had to say when Inspector Lestrade interrogated them about this.
   Alice: Emily or Felix stole the jewel.
   Bob: Charles or Felix stole the jewel.
   Charles: Alice is a knave.
   Debbie: Charles and Felix are knaves.
   Emily: I can claim that Debbie is a knight.
   Felix: Bob or Emily is a knight.

   Who stole the jewel?

   Solution: Emily. Note that if somebody says "I can claim that statement P is true", then P is indeed true irrespective of if that person is a knight or a knave. From Emily's statement we can conclude that Debbie is a knight (we don't know if Emily is a knight or knave yet). So both Charles and Felix are lying. Since Charles is lying, Alice is speaking the truth. Thus one of Emily or Felix stole the jewel. Since Felix is lying, Bob is a knave, so neither Charles nor Felix stole the jewel. Thus Emily stole the jewel.

3. Evaluate the following infinite sums (you are given that they are well defined finite numbers):

   1. $\sum_{k=1}^{\infty} \frac{k}{3^k}$
   2. $\sum_{k=1}^{\infty} \frac{k^2}{3^k}$

Solution: $\frac{3}{4}, \frac{3}{2}$.

1. Let $S := \sum_{k=1}^{\infty} \frac{k}{3^k}$. We have

$$3 \cdot S = \sum_{k=0}^{\infty} \frac{k+1}{3^k} = S + \sum_{k=0}^{\infty} \frac{1}{3^k} = S + \frac{3}{2}$$

   Thus $S = \frac{3}{4}$.

2. Let $T := \sum_{k=1}^{\infty} \frac{k^2}{3^k}$. We have

$$3 \cdot T = \sum_{k=0}^{\infty} \frac{(k+1)^2}{3^k} = T + 2S + \frac{3}{2}$$

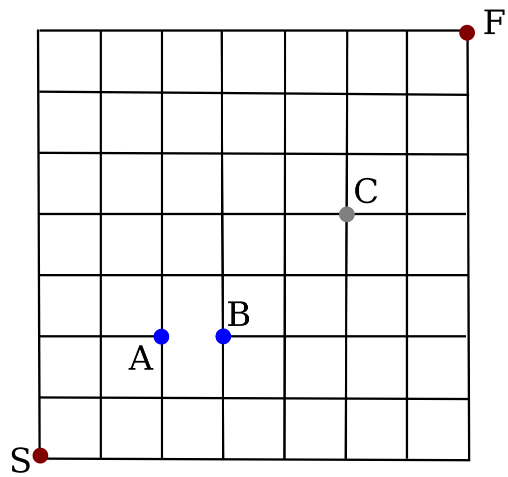
   Thus

$$2 \cdot T = \frac{3}{2} + \frac{3}{2}$$

   So $T = \frac{3}{2}$.

4. In how many ways can you go from start (S) to finish (F) using only ups and rights with the following constraints:

    1. cannot take path between A and B,
    2. cannot travel through C.



Solution: 1776. By the Inclusion-Exclusion principle, we have
number of such ways = total number of paths from S to F
− number of paths from S to F going through the edge AB
− number of paths from S to F going through the vertex C
+ number of paths from S to F going through the edge AB and going through the vertex C.
We can use the multiplication principle and the bijection principle to solve each of the subproblems.

$$\text{number of such ways} = \binom{14}{7} - \binom{4}{2}\cdot\binom{9}{4} - \binom{9}{5}\cdot\binom{5}{2} + \binom{4}{2}\cdot\binom{4}{2}\cdot\binom{5}{2}$$

$$= \binom{14}{7} - \binom{9}{5}\cdot(6+10) + 6\cdot 6\cdot 10 = 1776$$

5. Find all positive integer solutions to the equation $x^2 + y^2 = 2017$.

Solution: The solutions (x, y) are $(44, 9)$ and $(9, 44)$.
W.L.O.G. $x = 2x_1$ is even, and $y = 2y_1 + 1$ is odd. The problem reduces to solving $x_1^2 + y_1^2 + y_1 = 504$. It follows that $y_1 \pmod 4$ has to be 0 or 3, since perfect squares are either 0 or 1 modulo 4.
Let us consider the case that $y_1 \equiv 0 \pmod 4$, let $x_1 = 2x_2$, $y_1 = 4y_2$. The problem reduces to solving $x_2^2 + 4y_2^2 + y_2 = 126$. We have $4y_2^2 \leq 126$ thus y_2 has to be between 0 and 7 and among them we can see that $y_2 = 1$, $x_2 = 11$ is the only solution, which corresponds to the solution $(x, y) = (44, 9)$ of our original equation.
A similar arguement shows that there are no solution for $y_1 \equiv 3 \pmod 4$.

6. Suppose you climb Tech Tower, and you see three signposts for HSMC on Cherry Street with angles of depression $30°$, $45°$ and $60°$. You know that the consecutive signposts are at distance 30 meters apart. How high is Tech Tower?
Here we are assuming that Cherry Street is a straight line at ground level and the signposts are point objects on it. By angle of depression we mean the angle between the horizontal and the line of sight (i.e. the line joining you and the signpost).

Solution: $15\sqrt{6}$ meters. Let P and Q be the peak and base of Tech Tower, respectively. Let A, B and C be the three signposts corresponding to the angles of depression $30°$, $45°$ and $60°$. Let $h = PQ$ be the height of Tech Tower. Then we have $AQ = \sqrt{3}h$, $BQ = h$ and $CQ = \frac{h}{\sqrt{3}}$. By Appollonius's Theorem, we have $QA^2 + QC^2 = 2(QB^2 + AB^2)$ i.e. $h = AB \cdot \sqrt{\frac{2}{3+\frac{1}{3}-2}} = 30 \cdot \sqrt{\frac{3}{2}} = 15 \cdot \sqrt{6}$.

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