1. Find the smallest 4-digit number divisible by both 21 and 9.

**Solution:** 1008. The question is equivalent to finding smallest 4-digit number divisible by 7 and 9. Note that 1008 is the smallest 4-digit number divisible by 9, which is also divisible by 7.

2. What is the length of the interval where $4 \sin x - 3 \cos x$ can take values?

**Solution:** 10. We can write $4 \sin x - 3 \cos x = 5 \cdot \left( \frac{4}{5} \sin x - \frac{3}{5} \cos x \right) = 5 \sin(x + y)$ where $y$ is such that $\cos y = \frac{4}{5}$ and $\sin y = -\frac{3}{5}$. Now, as $x$ and $y$ vary over all real numbers, $5 \sin(x + y)$ takes all values in $[-5, 5]$. So the length of the interval is 10.

3. Two fair six-sided dice are rolled. What is the probability that the sum of the numbers rolled is a prime number?

**Solution:** $\frac{5}{12}$. There are $6^2 = 36$ possible pairs of numbers rolled. The sum of the two dice is an integer between 2 and 12 inclusive, so the possible prime numbers rolled are 2, 3, 5, 7, and 11. There is one way to roll a sum of 2, two ways to roll a sum of 3, four ways to roll a sum of 5, six ways to roll a sum of 7, and two ways to roll a sum of 11. Thus, the probability of rolling a sum that is prime is $\frac{1 + 2 + 4 + 6 + 2}{36} = \frac{15}{36} = \frac{5}{12}$.

4. Take three coins of radius 2 and arrange them in a triangle touching each other. What is the area of the region left in between the coins?

**Solution:** $4\sqrt{3} - 2\pi$. Area of region = Area of equilateral triangle of sides 4 - 3× (Area of sector of radius 2 and angle 60°) $= \frac{\sqrt{3}}{4} \cdot 4^2 - \frac{\sqrt{3}}{6} \cdot \pi \cdot 2^2 = 4\sqrt{3} - 2\pi$. 
5. Find the sum of squares of the roots of the polynomial

\[ x^7 - 3x^6 + 25x^5 - 34x^3 - 56x^2 + 17x + 2017 \]

**Solution:** \(-41\). Let \( \{ \alpha_i \} \) be the roots of the polynomial. We have

\[
\left( \sum \alpha_i^2 \right) = \left( \sum \alpha_i \right)^2 - 2 \left( \sum \alpha_i \alpha_j \right) = (3)^2 - 2 \cdot 25 = -41.
\]

6. What is the slope of the line tangent to the closest point to the origin on the circle

\[ x^2 + y^2 + 8x + 10y + 40 = 0 \]

**Solution:** \(-\frac{4}{5}\). Note that the center of the circle is \((-4, -5)\) and the line joining the origin to \((-4, -5)\) the circle has slope \(\frac{5}{4}\). By Euclidean geometry, this line must be perpendicular to the tangent at point on the circle closest to the origin. Thus, the slope of the tangent line is \(-\frac{4}{5}\).

7. Evaluate \(\log_{10} \cos \tan^{-1} 3\).

**Solution:** \(-\frac{1}{2}\). Consider a triangle \(\triangle ABC\) with \(\angle ABC = 90^\circ\), \(AB = 3\), \(BC = 1\) and \(CA = \sqrt{10}\). Then \(\tan^{-1} 3 = \angle BCA\) and \(\cos \angle BCA = BC/CA = \frac{1}{\sqrt{10}}\), now take logarithm.

8. What is the remainder when 12345678910111213 is divided by 101?

**Solution:** 47. Since 100 \(\equiv -1 \pmod{101}\), we have:

\[
12345678910111213 \equiv 1 + (-23 + 45) + (-67 + 89) + (-10 + 11) + (-12 + 13) \\
\equiv 1 + 22 + 22 + 1 + 1 \equiv 47 \pmod{101}.
\]

9. At St. Ives High School, each student takes 7 classes, and each class contains 7 students. Every pair of students have exactly 3 classes in common. How many students go to school in St. Ives?
**Solution:** 15. A student has 6 classmates in each of 7 classes, for a total of 42 classmates (counting repeats.) As each other student is counted a classmate 3 times, there are $\frac{42}{3} = 14$ other students, 15 in total. An example of such a configuration is given by the (15,7,3)-incidence graph, taking students/classes as the analogues of points/sets.

10. How many ordered triples \((a, b, c)\) of positive integers satisfy

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.
\]

**Solution:** 10. Assume that \(a \leq b \leq c\). Note that all of \(a, b, c\) must be at least 2. If \(a \geq 3\) then in fact all of \(a, b, c\) must also equal 3 since otherwise the left side would not be large enough to sum to 1. Otherwise, \(a = 2\) and the equation reduces to \(\frac{1}{b} + \frac{1}{c} = \frac{1}{2}\). If \(b \geq 4\) then both \(b\) and \(c\) must equal 4 since otherwise the left side would not be large enough to sum to \(\frac{1}{2}\). If \(b = 3\) then \(c = 6\), and there is no solution for \(b = 2\). Thus, all such triples are rearrangements of \((3, 3, 3)\), \((2, 4, 4)\), \((2, 3, 6)\), and there are \(1 + 3 + 6 = 10\) of them.

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