HSMC 2017 Cypher

1. Find the smallest 4-digit number divisible by both 21 and 9.

> **Solution:** 1008. The question is equivalent to finding smallest 4-digit number divisible by 7 and 9. Note that 1008 is the smallest 4-digit number divisible by 9, which is also divisible by 7.

2. What is the length of the interval where $4\sin x - 3\cos x$ can take values?

> **Solution:** 10. We can write $4\sin x - 3\cos x = 5 \cdot (\frac{4}{5}\sin x - \frac{3}{5}\cos x) = 5\sin(x+y)$ where y is such that $\cos y = \frac{4}{5}$ and $\sin y = -\frac{3}{5}$. Now, as x and y vary over all real numbers, $5\sin(x+y)$ takes all values in $[-5, 5]$. So the length of the interval is 10.

3. Two fair six-sided dice are rolled. What is the probability that the sum of the numbers rolled is a prime number?

> **Solution:** $\dfrac{5}{12}$. There are $6^2 = 36$ possible pairs of numbers rolled. The sum of the two dice is an integer between 2 and 12 inclusive, so the possible prime numbers rolled are 2, 3, 5, 7, and 11. There is one way to roll a sum of 2, two ways to roll a sum of 3, four ways to roll a sum of 5, six ways to roll a sum of 7, and two ways to roll a sum of 11. Thus, the probability of rolling a sum that is prime is $\dfrac{1+2+4+6+2}{6^2} = \dfrac{15}{36} = \dfrac{5}{12}$.

4. Take three coins of radius 2 and arrange them in a triangle touching each other. What is the area of the region left in between the coins?

> **Solution:** $4\sqrt{3} - 2\pi$. Area of region = Area of equilateral triangle of sides $4 - 3\times$ (Area of sector of radius 2 and angle 60°) $= \frac{\sqrt{3}}{4} \cdot 4^2 - \frac{3}{6} \cdot \pi \cdot 2^2 = 4\sqrt{3} - 2\pi$.

5. Find the sum of squares of the roots of the polynomial

$$x^7 - 3x^6 + 25x^5 - 34x^3 - 56x^2 + 17x + 2017$$

Solution: -41. Let $\{\alpha_i\}$ be the roots of the polynomial. We have

$$(\sum_i \alpha_i^2) = (\sum_i \alpha_i)^2 - 2(\sum_{i<j} \alpha_i\alpha_j) = (3)^2 - 2 \cdot 25 = -41.$$

6. What is the slope of the line tangent to the closest point to the origin on the circle $x^2 + y^2 + 8x + 10y + 40 = 0$?

Solution: $-\frac{4}{5}$. Note that the center of the circle is $(-4, -5)$ and the line joining the origin to $(-4, -5)$ the circle has slope $\frac{5}{4}$. By Euclidean geometry, this line must be perpendicular to the tangent at point on the circle closest to the origin. Thus, the slope of the tangent line is $-\frac{4}{5}$.

7. Evaluate $\log_{10} \cos \tan^{-1} 3$.

Solution: $-\frac{1}{2}$. Consider a triangle $\triangle ABC$ with $\angle ABC = 90°, AB = 3, BC = 1$ and $CA = \sqrt{10}$. Then $\tan^{-1} 3 = \angle BCA$ and $\cos \angle BCA = BC/CA = \frac{1}{\sqrt{10}}$, now take logarithm.

8. What is the remainder when 12345678910111213 is divided by 101?

Solution: 47. Since $100 \equiv -1 \pmod{101}$, we have:

$$12345678910111213 \equiv 1 + (-23 + 45) + (-67 + 89) + (-10 + 11) + (-12 + 13)$$

$$\equiv 1 + 22 + 22 + 1 + 1 \equiv 47 \pmod{101}.$$

9. At St. Ives High School, each student takes 7 classes, and each class contains 7 students. Every pair of students have exactly 3 classes in common. How many students go to school in St. Ives?

Solution: 15. A student has 6 classmates in each of 7 classes, for a total of 42 classmates (counting repeats.) As each other student is counted a classmate 3 times, there are $\frac{42}{3} = 14$ other students, 15 in total. An example of such a configuration is given by the (15,7,3)-incidence graph, taking students/classes as the analogues of points/sets.

10. How many ordered triples (a, b, c) of positive integers satisfy

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Solution: 10. Assume that $a \leq b \leq c$. Note that all of a, b, c must be at least 2. If $a \geq 3$ then in fact all of a, b, c must also equal 3 since otherwise the left side would not be large enough to sum to 1. Otherwise, $a = 2$ and the equation reduces to $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$. If $b \geq 4$ then both b and c must equal 4 since otherwise the left side would not be large enough to sum to $\frac{1}{2}$. If $b = 3$ then $c = 6$, and there is no solution for $b = 2$. Thus, all such triples are rearrangements of $(3, 3, 3), (2, 4, 4), (2, 3, 6)$, and there are $1 + 3 + 6 = 10$ of them.

Problems contributed by Jose Acevedo, Marcel Celaya, Santhosh Karnik, George Kerchev, Sudipta Kolay, Peter Ralli, Libby Taylor, Hagop Tossounian and Chi Ho Yuen.